On the motion of spinning test particles in plane gravitational waves

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Abstract

The Mathisson-Papapetrou-Dixon equations for a massive spinning test particle in plane gravitational waves are analysed and explicit solutions constructed in terms of solutions of certain linear ordinary differential equations. For harmonic waves this system reduces to a single equation of Mathieu-Hill type. In this case spinning particles may exhibit parametric excitation by gravitational fields. For a spinning test particle scattered by a gravitational wave pulse, the final energy-momentum of the particle may be related to the width, height, polarisation of the wave and spin orientation of the particle.

1 Introduction

One of the most striking predictions of general relativity is the existence of gravitational waves. Such waves are thought to be produced by astrophysical phenomena ranging from the coalescence of orbiting binaries to violent events in the early Universe. Their detection would herald a new window for the observation of natural phenomena[1]. An essential mechanism by which gravitational waves interact with matter relies on the tidal forces produced on objects with structure in a gravitational field. This mechanism is responsible for forces and torques experienced by extended bodies or elementary particles with angular momentum in curved spacetime. By neglecting self-gravitation and back-reaction the dynamics of classical test particles with angular momentum was first studied in detail by Mathisson, Fock, Papapetrou *et al* [2]. The theory was further clarified by Dixon [3] using a rationalised multipole expansion technique and developed by Ehlers, Rudolph [4] and others.

Up to the dipole approximation the motion of a classical spinning test particles is governed by the Mathisson-Papapetrou-Dixon (MPD) equations which encode a spin-curvature coupling that may play an important role in astrophysical contexts. In particular the MPD equations predict gravitational spin-spin interactions between rotating stars and orbiting massive spinning particles. The ever-increasing interest in black holes and the anticipation of gravitational wave astronomy have stimulated renewed interest in the dynamics of spinning particles near massive compact objects. This includes gravitational waves generated by spinning particles spiralling into black holes [6, 7, 8], chaotic behaviour of spinning particles in the Kerr metric [9, 10, 11] and numerical simulations for general orbital motions of spinning particles in stationary spacetime [12].

By contrast the dynamical response of spinning test particles to strong tidal forces produced by nearby gravitational events is less well understood although it had been suggested that the scattering of particles by plane gravitational waves provides a good local representation for such a physical process [13]. This problem has been recently considered from a number of different perspectives [14, 15, 16]. In this paper we offer a

new approach. We exploit the Killing symmetries of the plane gravitational wave metric and suitable spacetime charts to construct a class of solutions in terms of certain linear ordinary differential equations. For harmonic gravitational waves this system reduces to a single equation of Mathieu-Hill type. By analogy with the parametric excitation of dynamical systems it is suggested that nonlinear spin-curvature interactions may exhibit parametric excitation by gravitational fields. In the case of a spinning test particle scattered by a square gravitational wave pulse, the final energy-momentum of the particle is shown to depend on the width, height, polarisation of the wave and spin orientation of the particle relative to a geodesic observer.

2 Equations of Motion for Spinning Test Particles

In the monopole-dipole approximation the MPD equations determine the world-line of a spinning test particle with tangent vector V, momentum vector P and spin 2-form s in terms of the spacetime metric g, its Levi-Civita connection ∇ and Riemannian curvature tensor R. [1] By introducing the metric duals $p \equiv \widetilde{P} \equiv g(P,-)$ and $v \equiv \widetilde{V}$, denoting interior operator (contraction with any vector V) on forms by $\imath_V$ [17] the MPD equations [3] can be written in a compact form as

$$\nabla_V \, p \,=\, \imath_V \, f \tag{1}$$

$$\nabla_V \, s \,=\, 2 \, p \wedge v \tag{2}$$

with the spin condition [5]

$$\imath_P \, s \,=\, 0 \tag{3}$$

and forcing term

$$f \,=\, \frac{1}{4} \star^{-1}(R_{ab} \wedge \star \, s) \, e^a \wedge e^b \tag{4}$$

in terms of the exterior product $\wedge$ and Hodge map $\star$ associated with a metric with the signature $(-,+,+,+)$. Thus the "inverse Hodge map" denoted by $\star^{-1}$ [17] acts on any q-form ω according to $\star^{-1}\omega = -(-1)^{q(4-q)} \star\omega$ satisfying $\star(\star^{-1}\omega) = \star^{-1}(\star\omega) = \omega$. The curvature 2-forms $R^a{}_b$ in any basis $\{X_a\}$ with co-basis $\{e^a\}$ are related to the curvature tensor components $R^a{}_{bcd}$ by $2 \, R^a{}_b(X_c, X_d) = R^a{}_{bcd}$, $(a, b, c, d = 0, 1, 2, 3)$. In terms of the components $f_a = f(X_a)$, $v^a = e^a(V)$, $p^a = e^a(P)$, $s_{ab} = s(X_a, X_b)$ (4) becomes

$$f_a \,=\, -\frac{1}{2} \, R_{abcd} \, v^b \, s^{cd}.$$

It follows that [6] the velocity of the spinning test particle takes the following explicit form

$$v^a \,=\, \frac{p^c \, v_c}{p^f \, p_f} \left(p^a - \frac{2 \, R_{bcde} \, p^c \, s^{ab} \, s^{de}}{4 \, p^c \, p_c - R_{bcde} \, s^{bc} \, s^{de}} \right). \tag{5}$$

The freedom to normalise V ensures that $p^c \, v_c$ is arbitrary and once a parameterisation of the world-line has been chosen (5) permits the computation of the world-line [4]. The norm of the momentum vector given by

$$m \,=\, \sqrt{-g(P,P)} \tag{6}$$

may be identified as the mass of the particle. This is used to define the normalised momentum vector given by

$$U \,=\, \frac{P}{m} \tag{7}$$

[1] In this paper we choose units where $c = G = 1$.

In Minkowski spacetime U coincides with the 4-velocity $V/\sqrt{-g(V,V)}$ but in general these two vectors differ in curved spacetime. Using the 1-form $u=\widetilde{U}$ the spin 1-form l is defined by

$$l = -\frac{1}{2} \star (u \wedge s) \tag{8}$$

and spin vector by $L=\widetilde{l}$. The angular-momentum 2-form s may be related back to the spin 1-form by

$$s = 2\, \imath_U \star^{-1} l. \tag{9}$$

The norm of the spin vector L denoted

$$\ell = \sqrt{g(L,L)} \tag{10}$$

defines the "spin" of the particle. For convenience we introduce the "reduced" spin vector , $\Sigma = L/m$, and reduced spin 1-form $\sigma = l/m$ such that the norm $\alpha = \sqrt{g(\Sigma,\Sigma)} = \ell/m$ denotes the spin to mass ratio. It follows from (1), (2) and (3) that α and m are constants of motion and the vectors U and Σ are orthogonal, i.e. $g(U,\Sigma)=0$.

We are interested in solving the MPD equations in a gravitational wave spacetime that admits enough symmetry to permit the construction of constants along the world-line $\mathcal{C}$. If sufficient constants can be found they enable one to integrate the equations of motion in terms of known functions. Suppose the metric g admits a Killing vector field K and let $k=\widetilde{K}$. It follows from the defining equation $\mathcal{L}_K\, g = 0$ that

$$(\nabla_Y\, k)(Z) + (\nabla_Z\, k)(Y) = 0 \tag{11}$$

and

$$\nabla_Y\, \mathrm{d}k = 2\, k^a\, y^b\, R_{ab} \tag{12}$$

for any vectors Y and Z where $k^a = e^a(K)$ and $y^a = e^a(Y)$. The preceding relation is equivalent to

$$-k_{[c;d];a}\, y^a = k^a\, y^b\, R_{abcd}$$

in component notation. By introducing the 4-form

$$\gamma_K = p \wedge \star k + \frac{1}{4}\, \mathrm{d}\, k \wedge \star s = \left\{ k_a\, p^a - \frac{1}{2}\, k_{[a;b]}\, s^{ab} \right\} \star 1 \tag{13}$$

the MPD equations (1), (2) and (3) together with the above relations then imply that

$$\nabla_V\, \gamma_K = \nabla_V\, p \wedge \star k + p \wedge \star \nabla_V\, k + \frac{1}{4}\, \nabla_V\, (\mathrm{d}k) \wedge \star\, s + \frac{1}{4}\, \mathrm{d}k \wedge \star \nabla_V\, s = 0. \tag{14}$$

Therefore for spacetime with a Killing vector K the quantity

$$C = \star^{-1} \left\{ \widetilde{K} \wedge \star p + \frac{1}{4}\, \mathrm{d}\, \widetilde{K} \wedge \star s \right\} = k_a\, p^a - \frac{1}{2}\, k_{[a;b]}\, s^{ab} \tag{15}$$

is preserved along the world-line of a spinning test particle [18].

3 The MPD Equations in a Plane Gravitational Wave Metric

In the coordinate chart $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ we represent a plane gravitational wave metric g in the Kerr-Schild form [19]:

$$g = H(\hat{\mu}, \hat{x}, \hat{y})\, \mathrm{d}\hat{\mu} \otimes \mathrm{d}\hat{\mu} - \frac{1}{2}\,(\mathrm{d}\hat{\mu} \otimes \mathrm{d}\hat{\nu} + \mathrm{d}\hat{\nu} \otimes \mathrm{d}\hat{\mu}) + \mathrm{d}\hat{x} \otimes \mathrm{d}\hat{x} + \mathrm{d}\hat{y} \otimes \mathrm{d}\hat{y} \tag{16}$$

where $\hat{\mu} = \hat{t} - \hat{z}$ and $\hat{\nu} = \hat{t} + \hat{z}$ and these coordinates have dimension of length. We choose

$$H(\hat{\mu}, \hat{x}, \hat{y}) = f(\hat{\mu})\,(\hat{x}^2 - \hat{y}^2) \tag{17}$$

for an arbitrary wave profile $f(\hat{\mu})$ and adopt the orthonormal co-basis

$$\begin{aligned} e^0 &= \frac{\mathrm{d}\hat{\nu}}{2} + (1 - H(\hat{\mu}, \hat{x}, \hat{y}))\,\frac{\mathrm{d}\hat{\mu}}{2} \\ e^1 &= \mathrm{d}\hat{x} \\ e^2 &= \mathrm{d}\hat{y} \\ e^3 &= \frac{\mathrm{d}\hat{\nu}}{2} - (1 + H(\hat{\mu}, \hat{x}, \hat{y}))\,\frac{\mathrm{d}\hat{\mu}}{2} \end{aligned} \tag{18}$$

with dual basis $\{X_a\}$ such that $e^a(X_b) = \delta^a{}_b$. This basis is parallel along the geodesic observer $\mathcal{O} : \tau \mapsto (\hat{t}(\tau) = \tau, \hat{x} = 0, \hat{y} = 0, \hat{z} = 0)$, i.e. $\nabla X_a|_{\mathcal{O}} = 0$, and takes the form $\{X_0 = \partial_{\hat{t}}, X_1 = \partial_{\hat{x}}, X_2 = \partial_{\hat{y}}, X_3 = \partial_{\hat{z}}\}$ which sets up a "local Lorentz frame" along $\mathcal{O}$. In the coordinates $(\hat{\mu}, \hat{\nu}, \hat{x}, \hat{y})$ this observer is parametrised by $\mathcal{O} : \tau \mapsto (\hat{\mu}(\tau) = \tau, \hat{\nu}(\tau) = \tau, \hat{x} = 0, \hat{y} = 0)$.

It is instructive to study the motion of a spinning test particle with mass m, spin $\ell = \alpha\, m$ and initial location coinciding with the observer $\mathcal{O}$ at $\tau = 0$ excited from rest by the incident gravitational wave. Thus we parametrise the world-line of this particle by $\mathcal{C} : \lambda \mapsto (\hat{\mu}(\lambda) = \lambda, \hat{\nu}(\lambda), \hat{x}(\lambda), \hat{y}(\lambda))$ for functions $\hat{\nu}(\lambda), \hat{x}(\lambda), \hat{y}(\lambda)$ subject to the initial conditions:

$$\hat{\nu}(0) = \hat{x}(0) = \hat{y}(0) = 0 \tag{19}$$

Let $u^a = e^a(U)$ and $\sigma^a = e^a(\Sigma)$ be functions of λ satisfying

$$\eta_{ab}\, u^a(\lambda)\, u^b(\lambda) = -1 \tag{20}$$

$$\eta_{ab}\, \sigma^a(\lambda)\, \sigma^b(\lambda) = \alpha^2 \tag{21}$$

$$\eta_{ab}\, u^a(\lambda)\, \sigma^b(\lambda) = 0 \tag{22}$$

for $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The initial conditions for u^a are simply

$$u^0(0) = 1,\; u^1(0) = u^2(0) = u^3(0) = 0 \tag{23}$$

while the initial conditions for σ^a incorporate the initial spin orientation according to

$$\sigma^0(0) = 0,\; \sigma^1(0) = \alpha_1,\; \sigma^2(0) = \alpha_2,\; \sigma^3(0) = \alpha_3 \tag{24}$$

where constants α_1, α_2, α_3 satisfy $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \alpha^2$.

The Killing symmetry of g is more conveniently expressed in the coordinates (μ, ν, x, y) defined by [19]

$$\hat{\mu} = \mu \tag{25}$$

$$\hat{\nu} = \nu + x^2\, a(\mu)\, a'(\mu) + y^2\, b(\mu)\, b'(\mu) \tag{26}$$

$$\hat{x} = a(\mu)\, x \tag{27}$$

$$\hat{y} = b(\mu)\, y \tag{28}$$

where the metric functions $a(\mu)$ and $b(\mu)$ satisfy

$$a''(\mu) = f(\mu)\, a(\mu) \tag{29}$$

$$b''(\mu) = -f(\mu)\, b(\mu) \tag{30}$$

with a convenient choice of the (gauge fixing) conditions:

$$a(0) = b(0) = 1 \tag{31}$$

$$a'(0) = b'(0) = 0. \tag{32}$$

In this coordinate chart the metric g takes the Rosen form

$$g = -\frac{1}{2}\,(\mathrm{d}\mu \otimes \mathrm{d}\nu + \mathrm{d}\nu \otimes \mathrm{d}\mu) + a(\mu)^2\, \mathrm{d}x \otimes \mathrm{d}x + b(\mu)^2\, \mathrm{d}y \otimes \mathrm{d}y \tag{33}$$

admitting five Killing vectors [19]:

$$K_1 = \frac{\partial}{\partial \nu},\ K_2 = \frac{\partial}{\partial x},\ K_3 = \frac{\partial}{\partial y} \tag{34}$$

$$K_4 = A(\mu)\, \frac{\partial}{\partial x} + 2\, x \frac{\partial}{\partial \nu} \tag{35}$$

$$K_5 = B(\mu)\, \frac{\partial}{\partial y} + 2\, y \frac{\partial}{\partial \nu} \tag{36}$$

where the functions $A(\mu)$, $B(\mu)$ satisfy

$$A'(\mu) = \frac{1}{a(\mu)^2},\ B'(\mu) = \frac{1}{b(\mu)^2}. \tag{37}$$

For simplicity we set $A(0) = B(0) = 0$. It follows from (15), (19), (23) and (24) that

$$C_1 = -\frac{m}{2},\ C_2 = 0,\ C_3 = 0,\ C_4 = m\, \alpha_2,\ C_5 = -m\, \alpha_1. \tag{38}$$

From (19) and (25) – (28) in the coordinates (μ, ν, x, y) the initial conditions for the world-line of the spinning test particle $\mathcal{C} : \lambda \mapsto (\mu(\lambda) = \lambda, \nu(\lambda), x(\lambda), y(\lambda))$ read

$$\nu(0) = x(0) = y(0) = 0 \tag{39}$$

Thus (20) and (38) yield

$$u^0(\lambda) = 1 + \frac{\alpha_2^2}{2}\, a'(\lambda)^2 + \frac{\alpha_1^2}{2}\, b'(\lambda)^2 \tag{40}$$

$$u^1(\lambda) = -\alpha_2\, a'(\lambda) \tag{41}$$

$$u^2(\lambda) = \alpha_1\, b'(\lambda) \tag{42}$$

$$u^3(\lambda) = \frac{\alpha_2^2}{2}\, a'(\lambda)^2 + \frac{\alpha_1^2}{2}\, b'(\lambda)^2 \tag{43}$$

while (22) and (38) give

$$\sigma^0(\lambda) = \frac{1}{\alpha_2^2\, a'(\lambda)^2 + \alpha_1^2\, b'(\lambda)^2 - 2} \left\{ \left(\alpha_2^2\, a'(\lambda)^2 + \alpha_1^2\, b'(\lambda)^2\right) \sigma^3(\lambda) \right.$$

$$\left. - 2(y(\lambda) - \alpha_1)\alpha_2 b(\lambda) a'(\lambda) - 2(x(\lambda) + \alpha_2)\alpha_1 a(\lambda) b'(\lambda) \right\} \tag{44}$$

$$\sigma^1(\lambda) = -(y(\lambda) - \alpha_1) b(\lambda) + \frac{1}{\alpha_2^2\, a'(\lambda)^2 + \alpha_1^2\, b'(\lambda)^2 - 2} \left\{ -2\alpha_2\, a'(\lambda) \sigma^3(\lambda) \right.$$

$$\left. +2\,\alpha_2\, a'(\lambda) \left[(y(\lambda) - \alpha_1)\alpha_2 b(\lambda) a'(\lambda) + (x(\lambda) + \alpha_2)\alpha_1 a(\lambda) b'(\lambda) \right] \right\} \tag{45}$$

$$\sigma^2(\lambda) = (x(\lambda) + \alpha_2) a(\lambda) + \frac{1}{\alpha_2^2\, a'(\lambda)^2 + \alpha_1^2\, b'(\lambda)^2 - 2} \left\{ 2\alpha_1\, b'(\lambda) \sigma^3(\lambda) \right.$$

$$\left. -2\,\alpha_1\, b'(\lambda) \left[(y(\lambda) - \alpha_1)\alpha_2 b(\lambda) a'(\lambda) + (x(\lambda) + \alpha_2)\alpha_1 a(\lambda) b'(\lambda) \right] \right\}. \tag{46}$$

Furthermore (21) and (38) imply quadratic equation for $\sigma^3(\lambda)$:

$$\sigma^3(\lambda)^2 + \mathcal{P}(\lambda)\, \sigma^3(\lambda) + \mathcal{Q}(\lambda) = 0 \tag{47}$$

where

$$\mathcal{P}(\lambda) = -2\,(y(\lambda) - \alpha_1)\alpha_2 b(\lambda) a'(\lambda) - 2\,(x(\lambda) + \alpha_2)\alpha_1 a(\lambda) b'(\lambda) \tag{48}$$

and

$$\mathcal{Q}(\lambda) = -\frac{\alpha^2}{4} \left\{ \alpha_2^2\, a'(\lambda)^2 + \alpha_1^2\, b'(\lambda)^2 - 2 \right\}^2$$

$$+\frac{(y(\lambda) - \alpha_1)^2 b(\lambda)^2}{4} \left\{ 2\alpha_1^2\alpha_2^2 a'(\lambda)^2 b'(\lambda)^2 + \alpha_2^4 a'(\lambda)^4 + \alpha_1^4 b'(\lambda)^4 - 4\alpha_1^2 b'(\lambda)^2 + 4 \right\}$$

$$+\frac{(x(\lambda) + \alpha_2)^2 a(\lambda)^2}{4} \left\{ 2\alpha_1^2\alpha_2^2 a'(\lambda)^2 b'(\lambda)^2 + \alpha_2^4 a'(\lambda)^4 + \alpha_1^4 b'(\lambda)^4 - 4\alpha_2^2 a'(\lambda)^2 + 4 \right\}$$

$$+2\,\alpha_1\alpha_2 a(\lambda) b(\lambda) a'(\lambda) b'(\lambda)(x(\lambda) + \alpha_2)(y(\lambda) - \alpha_1). \tag{49}$$

The differential equations for $\nu(\lambda), x(\lambda), y(\lambda)$ are obtained from substituting (7), (9) and (40) – (46) into (5) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} x(\lambda) = 2\, f(\lambda) \left\{ \alpha_1 a(\lambda) b(\lambda) b'(\lambda)(x(\lambda) + \alpha_2)(y(\lambda) - \alpha_1) + \alpha_2 b(\lambda)^2 a'(\lambda)(y(\lambda) - \alpha_1)^2 \right.$$

$$\left. -(y(\lambda) - \alpha_1) b(\lambda) \sigma^3(\lambda) \right\} / \left\{ a(\lambda) \left(\alpha_2^2\, a'(\lambda)^2 + \alpha_1^2\, b'(\lambda)^2 - 2 \right) \right\} - \frac{(x(\lambda) + \alpha_2) a'(\lambda)}{a(\lambda)} \tag{50}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} y(\lambda) = 2\, f(\lambda) \left\{ \alpha_2 a(\lambda) b(\lambda) a'(\lambda)(x(\lambda) + \alpha_2)(y(\lambda) - \alpha_1) + \alpha_1 a(\lambda)^2 b'(\lambda)(x(\lambda) + \alpha_2)^2 \right.$$

$$\left. -(x(\lambda) + \alpha_2) a(\lambda) \sigma^3(\lambda) \right\} / \left\{ b(\lambda) \left(\alpha_2^2\, a'(\lambda)^2 + \alpha_1^2\, b'(\lambda)^2 - 2 \right) \right\} - \frac{(y(\lambda) - \alpha_2) b'(\lambda)}{b(\lambda)} \tag{51}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\nu(\lambda) =\ & 1 + (x(\lambda)+\alpha_2)^2 a'(\lambda)^2 + (y(\lambda)-\alpha_1)^2 b'(\lambda)^2 \\
& + f(\lambda)\left\{4\sigma^3(\lambda)(x(\lambda)+\alpha_2)(y(\lambda)-\alpha_1)(a(\lambda)b'(\lambda)+b(\lambda)a'(\lambda))\right. \\
& + 2\,(x(\lambda)+\alpha_2)^2 a(\lambda)^2\left(2-\alpha_2^2\,a'(\lambda)^2+\alpha_1^2\,b'(\lambda)^2-2\alpha_1\,b'(\lambda)^2 y(\lambda)\right) \\
& - 2\,(y(\lambda)-\alpha_1)^2 b(\lambda)^2\left(2+\alpha_2^2\,a'(\lambda)^2-\alpha_1^2\,b'(\lambda)^2+2\alpha_2\,a'(\lambda)^2 x(\lambda)\right) \\
& \left. - 4\,a(\lambda)b(\lambda)a'(\lambda)b'(\lambda)(x(\lambda)+\alpha_2)(y(\lambda)-\alpha_1)(\alpha_1 x(\lambda)+\alpha_2 y(\lambda))\right\} \\
& /\left(\alpha_2^2\,a'(\lambda)^2+\alpha_1^2\,b'(\lambda)^2-2\right). \qquad (52)
\end{aligned}
$$

Note that $\nu(\lambda)$ is obtained by directly integrating the R.H.S of (52) which is in turn decoupled from (50) and (51). Thus the world-line of the spinning particle is effectively determined by solving two coupled non-linear differential equations (50) and (51).

Given the initial reduced spin vector components α_1, α_2, α_3 with respect to the orthonormal basis $\{X_a\}$ along $\mathcal{C}$ the evolution of the normalised momentum vector U is readily available from the explicit expressions (40) – (43) for some gravitational wave profile $f(\mu)$ and the corresponding $a(\mu)$ and $b(\mu)$ (cf. (29), (30), (31), (32)). However to obtain the world-line $\mathcal{C}$ the differential equations (50), (51), (52) must be solved together with (47) subject to initial conditions (39) and the evolution of the reduced spin vector Σ follows from (44) – (47).

4 Solutions with Parallel Spin Vectors

A class of trivial solutions is obtained if the spin vector is initially in the direction of the incident gravitational wave ($\alpha_1 = \alpha_2 = 0, \alpha = |\alpha_3|$). In this case the world-line $\mathcal{C}$ stays coincident with the geodesic observer $\mathcal{O}$ while both U and Σ remain covariantly constant along the world-line. A class of non-trivial solutions describing non-geodesic motion of a particle with a spin vector which stays parallel along its world-line and is transverse to the propagation direction of the gravitational wave may be obtained as follows: Suppose $\alpha = |\alpha_1|$, $\alpha_2 = \alpha_3 = 0$ and

$$x(\lambda) = 0,\ \sigma^3(\lambda) = 0 \qquad (53)$$

then (50) is immediately satisfied and (51) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}y(\lambda) = \frac{(\alpha_1 - y(\lambda))b'(\lambda)}{b(\lambda)} \qquad (54)$$

which has a solution

$$y(\lambda) = \alpha_1\left(1 - \frac{1}{b(\lambda)}\right) \qquad (55)$$

compatible with the initial condition $y(0) = 0$. Equation (52) then reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\nu(\lambda) = 1 + \alpha_1^2\left(2\,f(\lambda) + \frac{b'(\lambda)^2}{b(\lambda)^2}\right) = 1 + \alpha_1^2\left(f(\lambda) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{b'(\lambda)}{b(\lambda)}\right) \qquad (56)$$

where (30) has been used. With the initial condition $\nu(0) = 0$ this can be integrated to yield

$$\nu(\lambda) = \lambda + \alpha_1^2\left(F(\lambda) - \frac{b'(\lambda)}{b(\lambda)}\right) \qquad (57)$$

where

$$F(\lambda) = \int_0^\lambda f(\eta)\mathrm{d}\eta. \tag{58}$$

Substituting the above equations into (40) – (46) we have

$$u^0(\lambda) = 1 + \frac{\alpha_1^2}{2} b'(\lambda)^2,\ u^1(\lambda) = 0,\ u^2(\lambda) = \alpha_1\, b'(\lambda),\ u^3(\lambda) = \frac{\alpha_1^2}{2}\, b'(\lambda)^2 \tag{59}$$

$$\sigma^0(\lambda) = 0,\ \sigma^1(\lambda) = \alpha_1,\ \sigma^2(\lambda) = 0,\ \sigma^3(\lambda) = 0. \tag{60}$$

Using (25) – (28) solutions (53), (55), (57) in the coordinates $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ are given by

$$\hat{t}(\lambda) = \lambda + \frac{\alpha_1^2}{2}\left\{(b(\lambda) - 2)\, b'(\lambda) + F(\lambda)\right\} \tag{61}$$

$$\hat{x}(\lambda) = 0 \tag{62}$$

$$\hat{y}(\lambda) = \alpha_1\,(b(\lambda) - 1) \tag{63}$$

$$\hat{z}(\lambda) = \frac{\alpha_1^2}{2}\, \left\{(b(\lambda) - 2)\, b'(\lambda) + F(\lambda)\right\}. \tag{64}$$

A further class of solutions describing non-geodesic motion is associated with the choice $\alpha_1 = \alpha_3 = 0$ $\alpha = |\alpha_2|$ and may be derived in a similar way:

$$u^0(\lambda) = 1 + \frac{\alpha_2^2}{2} a'(\lambda)^2,\ u^1(\lambda) = -\alpha_2\, a'(\lambda),\ u^2(\lambda) = 0,\ u^3(\lambda) = \frac{\alpha_2^2}{2}\, a'(\lambda)^2 \tag{65}$$

$$\sigma^0(\lambda) = 0,\ \sigma^1(\lambda) = 0,\ \sigma^2(\lambda) = \alpha_2,\ \sigma^3(\lambda) = 0 \tag{66}$$

with the corresponding world-line given by

$$\hat{t}(\lambda) = \lambda + \frac{\alpha_2^2}{2}\left\{(a(\lambda) - 2)\, a'(\lambda) - F(\lambda)\right\} \tag{67}$$

$$\hat{x}(\lambda) = -\alpha_2\,(a(\lambda) - 1) \tag{68}$$

$$\hat{y}(\lambda) = 0 \tag{69}$$

$$\hat{z}(\lambda) = \frac{\alpha_2^2}{2}\, \left\{(a(\lambda) - 2)\, a'(\lambda) - F(\lambda)\right\} \tag{70}$$

in the coordinates $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$.

The above solutions are valid for arbitrary wave profile $f(\mu)$ and the corresponding functions $a(\mu)$, $b(\mu)$, $F(\mu)$ satisfying (29), (30), (31), (32) and (58). For example a harmonic gravitational wave with angular frequency ω, dimensionless amplitude h and a mean value $k/2$ may be represented by

$$f(\mu) = \omega^2\, \left\{\frac{k}{4} - \frac{h}{2}\, \cos(\omega\, \mu)\right\}. \tag{71}$$

Introducing the variable $\zeta = \omega\, \mu/2$ one immediately sees that (29), (30) can be cast into the canonical form of the Mathieu equations [20]:

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\, a\left(\frac{2\zeta}{\omega}\right) = \{k - 2\, h \cos(2\,\zeta)\}\ a\left(\frac{2\zeta}{\omega}\right) \tag{72}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\, b\left(\frac{2\zeta}{\omega}\right) = \{-k + 2\, h \cos(2\,\zeta)\}\ b\left(\frac{2\zeta}{\omega}\right). \tag{73}$$

Thus $a(\mu)$ and $b(\mu)$ may be expressed in terms of Mathieu functions with characteristic numbers $\pm k$ and moduli $\pm h$ respectively. This suggests that nonlinear spin-curvature interactions may exhibit parametric excitation by gravitational fields [21] in analogy with the parametric excitation of dynamical systems.

5 Scattering by a Gravitational Wave Pulse

We now consider a spinning test particle scattered by a square gravitational wave pulse represented by the metric (33) with metric functions $a(\mu)$ and $b(\mu)$ associated with the wave profile $f(\mu)$:

$$f(\mu) = \begin{cases} 0 & \text{if } \mu < 0 \\ 1/\mathcal{L}^2 & \text{if } 0 \le \mu \le \delta \\ 0 & \text{otherwise.} \end{cases} \tag{74}$$

Since both the width δ and height parameter $\mathcal{L}$ carry dimension of length it is convenient to choose a length unit in which $\mathcal{L}$ is unity. To recover from this non-dimensionalisation one simply applies the substitutions: $\lambda \to \lambda/\mathcal{L}$, $\mu \to \mu/\mathcal{L}$, $\alpha \to \alpha/\mathcal{L}$, $m \to m/\mathcal{L}$, etc. With $\mathcal{L} = 1$ it follows from (29), (30), (31), (32) and (74) that

$$a(\mu) = \begin{cases} 1 & \text{if } \mu < 0 \\ \cosh(\mu) & \text{if } 0 \le \mu \le \delta \\ \sinh(\delta)\,\mu - \sinh(\delta)\,\delta + \cosh(\delta) & \text{otherwise} \end{cases} \tag{75}$$

$$b(\mu) = \begin{cases} 1 & \text{if } \mu < 0 \\ \cos(\mu) & \text{if } 0 \le \mu \le \delta \\ -\sin(\delta)\,\mu + \sin(\delta)\,\delta + \cos(\delta) & \text{otherwise} \end{cases} \tag{76}$$

As is well known such a sandwich wave gives rise to two half Minkowski spacetimes namely $\mathcal{M}_-$ for $\mu < 0$ and $\mathcal{M}_+$ for $\mu > \delta$ corresponding to the domains before and after the passage of the gravitational wave pulse. Within each half Minkowski spacetime the orthonormal basis $\{X_0 = \partial_{\hat{t}}, X_1 = \partial_{\hat{x}}, X_2 = \partial_{\hat{y}}, X_3 = \partial_{\hat{z}}\}$ defines a "global" Lorentz frame, whereas $\{X_a\}$ along the geodesic observer $\mathcal{O}$ still remains a local Lorentz basis in the "sandwiched" domain $\mathcal{D}$ where $0 \le \mu = \hat{t} - \hat{z} \le \delta$. It is therefore interesting to analyse how a spinning test particle initially following $\mathcal{O}$ with a spin orientation with respect to $\{X_a\}$ is scattered by the wave pulse and establish the final U and Σ. In $\mathcal{M}_-$ and $\mathcal{M}_+$ these vectors have different constant components u^a, σ^a and changes in these quantities along $\mathcal{C}$ across the domain $\mathcal{D}$ represent the gain in energy-momentum and precession of spin in the Lorentz frame associated with the observer $\mathcal{O}$. To establish the relations describing these changes consider the evolution of the components of U which is readily evaluated from (40) – (43) to be

Domain:	$\mathcal{M}_-$ $(\lambda < 0)$	$\mathcal{D}$ $(0 \le \lambda \le \delta)$	$\mathcal{M}_+$ $(\lambda > \delta)$	
$u^0(\lambda)$	1	$1 + \frac{\alpha_2^2}{2}\sinh^2(\lambda) + \frac{\alpha_1^2}{2}\sin^2(\lambda)$	$1 + \frac{\alpha_2^2}{2}\sinh^2(\delta) + \frac{\alpha_1^2}{2}\sin^2(\delta)$	
$u^1(\lambda)$	0	$-\alpha_2\sinh(\lambda)$	$-\alpha_2\sinh(\delta)$	(77)
$u^2(\lambda)$	0	$-\alpha_1\sin(\lambda)$	$-\alpha_1\sin(\delta)$	
$u^3(\lambda)$	0	$\frac{\alpha_2^2}{2}\sinh^2(\lambda) + \frac{\alpha_1^2}{2}\sin^2(\lambda)$	$\frac{\alpha_2^2}{2}\sinh^2(\delta) + \frac{\alpha_1^2}{2}\sin^2(\delta)$	

using (75) and (76) without explicitly solving (50) – (52). Since in Minkowski spacetime the normalised vector U coincides with the particle's 4-velocity, for $\lambda > \delta$ the motion of the particle in $\mathcal{M}_+$ has a 4-velocity with constant components $u^a(\lambda) = u^a(\delta)$ given in (77). Thus the difference between these values and the initial components $u^0 = 1, u^1 = u^2 = u^3 = 0$ determines the deflection of the particle motion relative to the observer $\mathcal{O}$.

To obtain the evolution of the reduced spin vector it is necessary to solve (50) – (52). If the initial reduced spin vector is perpendicular to the propagation direction of the wave then as described in the previous section these equations yield solutions (59) – (64) and (65) – (70) where $a(\mu)$, $b(\mu)$ are now fixed by (75), (76). For general α_1, α_2, α_3, a power series solution of (50) – (52) for $0 \leq \lambda \leq \delta$ may be obtained and from (44) – (47), (25) – (28) together with matching the particle trajectory $\mathcal{C}$ across domains $\mathcal{M}_-$, $\mathcal{D}$, $\mathcal{M}_+$ this yields

Domain:	$\mathcal{M}_-$ $(\lambda < 0)$	$\mathcal{D}$ $(0 \leq \lambda \leq \delta)$	$\mathcal{M}_+$ $(\lambda > \delta)$
$\hat{t}(\lambda)$	λ	$\left(1 + \alpha_1^2 - \alpha_2^2\right)\lambda + O(\lambda^3)$	$\hat{t}(\delta) + u^0(\delta)\,(\lambda - \delta)$
$\hat{x}(\lambda)$	0	$-\alpha_1\,\alpha_3\,\lambda + \dfrac{\alpha_2}{2}\,(\alpha^2 - 3\alpha_1^2 - \alpha_2^2 - 1)\,\lambda^2 + O(\lambda^3)$	$\hat{x}(\delta) + u^1(\delta)\,(\lambda - \delta)$
$\hat{y}(\lambda)$	0	$\alpha_2\,\alpha_3\,\lambda - \dfrac{\alpha_1}{2}\,(\alpha^2 - 3\alpha_2^2 - \alpha_1^2 - 1)\,\lambda^2 + O(\lambda^3)$	$\hat{y}(\delta) + u^2(\delta)\,(\lambda - \delta)$
$\hat{z}(\lambda)$	0	$\left(\alpha_1^2 - \alpha_2^2\right)\lambda + O(\lambda^3)$	$\hat{z}(\delta) + u^3(\delta)\,(\lambda - \delta)$

(78)

and

Domain:	$\mathcal{M}_-$ $(\lambda < 0)$	$\mathcal{D}$ $(0 \leq \lambda \leq \delta)$	$\mathcal{M}_+$ $(\lambda > \delta)$
$\sigma^0(\lambda)$	0	$-2\,\alpha_1\,\alpha_2\,\lambda + \dfrac{\alpha_3}{2}\,(\alpha_1^2 + \alpha_2^2)\,\lambda^2 + O(\lambda^3)$	$\sigma^0(\delta)$
$\sigma^1(\lambda)$	α_1	$\alpha_1 + \dfrac{\alpha_1}{2}\,(\alpha^2 - \alpha_1^2 + \alpha_2^2)\,\lambda^2 + O(\lambda^3)$	$\sigma^1(\delta)$
$\sigma^2(\lambda)$	α_2	$\alpha_2 + \dfrac{\alpha_2}{2}\,(\alpha^2 + \alpha_1^2 - \alpha_2^2)\,\lambda^2 + O(\lambda^3)$	$\sigma^2(\delta)$
$\sigma^3(\lambda)$	α_3	$\alpha_3 - \dfrac{\alpha_3}{2}\,(\alpha_1^2 + \alpha_2^2)\,\lambda^2 + O(\lambda^3)$	$\sigma^3(\delta)$

(79)

As expected for $\lambda > \delta$ the "out-going" motion of the particle in $\mathcal{M}_+$ follows a geodesic with constant 4-velocity components $u^a(\delta)$ given in (77) and constant reduced spin vector components $\sigma^a(\delta)$ (given approximately in (79)). This represents the precession of the spinning particle due to the scattering by the gravitational wave pulse.

6 Conclusions

We have considered the effects of a plane gravitational wave spacetime on the motion of a spinning test particle. A class of solutions of the MPD equations has been constructed in terms of solutions of certain linear ordinary differential equations. The motion of the particle with a parallel spin vector transverse to the direction of propagation of a polarised harmonic gravitational wave has been formulated in terms of Mathieu functions. By analogy with the parametric excitation of dynamical systems it is suggested that nonlinear spin-curvature interactions may exhibit parametric excitation by gravitational fields. For a spinning test particle scattered by a gravitational wave pulse the out-going energy-momentum of the particle has been expressed in terms of the width, height, polarisation of the wave and spin orientation of the particle relative to a geodesic observer.

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